PROBLEM 1

If Z is an arbitrary function of the variable quantities x and y and quantities involving their differentials p, q, r, s etc. so that its differential is of this kind

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + etc.,$$

to find the differential variation of the integral formula $\int Zdx$ *extended from the limit* $x = 0$ *to* $x = a$.

SOLUTION

Therefore, one has to find $\delta \int Zdx$, and since $\delta \int Zdx = \int \delta Zdx$, because of $\delta x = 0$, we will immediately have

$$\delta Z = N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

But for constant differential dx

$$\begin{aligned}
\delta p &= \frac{\delta dy}{dx} = \frac{d\delta y}{dx}, \\
\delta q &= \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}, \\
\delta r &= \frac{\delta dq}{dx} = \frac{d\delta q}{dx} = \frac{d^3\delta y}{dx^3}, \\
\delta s &= \frac{\delta dr}{dx} = \frac{d\delta r}{dx} = \frac{d^4\delta y}{dx},
\end{aligned}$$

whence we will obtain:

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{d^2\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + S\frac{d^4\delta y}{dx} + \text{etc.}$$

But now for the integration of the formula $\int \delta Zdx$, by integration by parts, we see that

$$\begin{aligned}
\int N\delta ydx &= \int \delta ydx \cdot N, \\
\int Pd\delta y &= P\delta y - \int \delta ydP, \\
\int Q\frac{dd\delta y}{dx} &= Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}dQ + \int \frac{\delta y}{dx}ddQ,
\end{aligned}$$

$$\int R\frac{d^3\delta y}{dx^2} = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2}dR + \frac{\delta y}{dx^2}ddR - \int \frac{\delta y}{dx^2}d^3R,$$
$$\int S\frac{d^4\delta y}{dx^3} = S\frac{d^3\delta y}{dx^3} - \frac{dd\delta y}{dx^3}dS + \frac{d\delta y}{dx^3}ddS - \frac{\delta y}{dx^3}d^3S - \int \frac{\delta y}{dx^3}d^4S$$
$$\text{etc.}$$

From these one hence derives the differential variation in question:

$$\begin{aligned}\int Zdx = \int \delta y dx \left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)\\ + \delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)\\ + \frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)\\ + \frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)\\ + \frac{d^3\delta y}{dx^3}\left(S - \text{etc.}\right)\\ + \text{etc.,}\end{aligned}$$

where the first part must be extended from the limit $x = 0$ to $x = a$, which therefore contains all intermediate variations; but in the remaining absolute parts one can immediately put $x = a$, and δy will denote the increment of the most outer value of y; but $d\delta y$, $dd\delta y$ etc. will additionally depend on the increments of the contiguous values.

COROLLARY 1

Therefore, if the integral formula $\int Zdx$ must be a maximum or a minimum for the limit $x = a$, it is necessary that its differential variation vanishes, no matter how the variations δy are assumed. Therefore, first it is clear that for all intermediate values of x

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

in which equation the required relation among x and y is contained.

COROLLARY 2

But hence, if the terms P, Q, R etc. occur, because of the integrations to be done, the relation among x and y is not determined completely, since through each integration an arbitrary constant is introduced into it. Therefore, in these cases one can add several other conditions to the question of the maxima or minimum, as, e.g., that for given values of x the other variable y has given values.

COROLLARY 3

But having omitted conditions of this kind, one can formulate a new question, how those constants introduced by integration must be defined in order to obtain either the maximum of all maxima or the minima of all minima: But for this it is necessary that for $x = a$ these equations are satisfied:

$$\begin{aligned}
&P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} = 0,\\
&Q - \frac{dR}{dx} + \frac{ddS}{dx^2} = 0,\\
&R - \frac{dS}{dx} = 0,\\
&S = 0.
\end{aligned}$$

COROLLARY 4

Further, for the same reasons it is necessary that for the other limit $x = 0$ these same equations are satisfied. For, since the differential variation must vanish for $x = 0$, the integral part will involve a constant which fulfills this condition; but this constant must render the absolute terms, if one puts $x = 0$ in them, equal to zero. Hence those formulas

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \text{etc.}, \quad Q - \frac{dR}{dx} + \text{etc.}, \quad R - \frac{dS}{dx} + \text{etc.}$$

must vanish in the case $x = 0$ and in the case $x = a$.

PROBLEM 2

If the function Z, aside from the quantities x, y, p, q etc., also contains the integral formula $\Phi = \int \mathfrak{Z} dx$ *somehow so that*

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds + etc.,$$

but in the formula Φ $\mathfrak{Z}$ *is an arbitrary function of x, y, p, q, r etc. while*

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + etc.,$$

to define the differential variation of the integral formula $\int Zdx$ *extended from the limit* $x = 0$ *to* $x = a$.

SOLUTION

Since $\delta \int Zdx = \int \delta Zdx$, as before, let us especially find δZ, and first it is immediately clear that

$$\delta Z = L\delta\Phi + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

where one, as before, has

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{d^3\delta y}{dx^3}, \quad \delta s = \frac{d^4\delta y}{dx^4} \quad \text{etc.},$$

but, because of $\delta\Phi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$, in like manner it will be

$$\delta\mathfrak{Z} = \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

and hence

$$\delta\Phi = \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

Therefore, since $Ldx\delta\Phi$ is the first term of the formula δZdx, it will be

$$\int Ldx\delta\Phi = \int Ldx \int \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right) dx.$$

Let $\int Ldx = V$ and one will have

$$\int Ldx\delta\Phi = V \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathrm{R}\delta r + \text{etc.}\right)$$
$$- \int Vdx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathrm{R}\delta r + \text{etc.}\right).$$

But here it does not matter, according to which law the integral $\int Ldx = V$ is taken; for, we would add a constant, and it would be cancelled in this expression again. Therefore, let us put this integral to be taken in such a way that it vanishes for $x = 0$, and since the differential variation must be accommodated to the limit $x = a$, it will be

$$\int Ldx\delta\Phi = -\int Vdx\,(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.})\,,$$

if to which the remaining parts are added, we conclude

$$\delta\int Zdx = \int dx(N - V\mathfrak{N})\delta y + (P - V\mathfrak{P})\delta p + (Q - V\mathfrak{Q})\delta q + \text{etc.},$$

where, if we apply the reduction indicated above, this differential variation already restricted to the limit $x = a$:

$$\begin{aligned}
&\int \delta y dx \left((N - V\mathfrak{N}) - \frac{d(P - V\mathfrak{P})}{dx} + \frac{dd(Q - V\mathfrak{Q})}{dx^2} - \frac{d^3(R - V\mathfrak{R})}{dx^3} + \text{etc.} \right) \\
&+ \delta y \left((P - V\mathfrak{P}) - \frac{d(Q - V\mathfrak{Q})}{dx} + \frac{dd(R - V\mathfrak{R})}{dx^2} - \text{etc.} \right) \\
&+ \frac{d\delta y}{dx} \left((Q - V\mathfrak{Q}) - \frac{d(R - V\mathfrak{R})}{dx^2} + \text{etc.} \right) \\
&+ \frac{dd\delta y}{dx^2} \left((R - V\mathfrak{R}) - \text{etc.} \right),
\end{aligned}$$

the structure of which expression is obvious.

Corollary 1

Therefore, this solution results from the preceding, if for the simple quantities N, P, Q, R etc. these composite ones are substituted:

$$N - V\mathfrak{N}, \quad P - V\mathfrak{P}, \quad Q - V\mathfrak{Q}, \quad R - V\mathfrak{R} \quad \text{etc.},$$

where $V = \int Ldx$, having taken the integral in such a way that it vanishes for $x = a$.

COROLLARY 2

Therefore, if the integral formula $\int Zdx$ must be rendered maximal or minimal for the limit $x = a$, one has to arrange that from the variation of all intermediate values of y no differential variation results, whence the relation among x and y is defined in such a way that:

$$(N - V\mathfrak{N}) - \frac{d(P - V\mathfrak{P})}{dx} + \frac{dd(Q - V\mathfrak{Q})}{dx^2} - \frac{d^3(R - V\mathfrak{R})}{dx^3} + \text{etc.} = 0,$$

which relation therefore already involves the prescribed limit $x = a$ so that, if another limit is prescribed, also another indefinite relation among x and y would result, since the quantity V also contains the value $x = a$.

COROLLARY 3

This way a relation among x and y of such a kind is found, from which the formula $\int Zdx$ obtains a maximum or minimum value in such a way that, while the most outer values of y remain the same, no matter how the intermediate values are changed, the value of the formula $\int Zdx$ will always result either smaller in the case of a maximum or larger in the case of a minimum than if the correct relation would be used.

COROLLARY 4

But if even the most outer values are permitted in our determination, from the differential variation we found one can also define those. Of course, the found relation must be determined by integrations in such a way that for $x = a$ also the absolute part vanishes. Hence one has to arrange that for $x = a$

$$(P - V\mathfrak{P}) - \frac{d(Q - V\mathfrak{Q})}{dx} + \frac{dd(R - V\mathfrak{R})}{dx^2} - \text{etc.} = 0,$$

$$(Q - V\mathfrak{Q}) - \frac{d(R - V\mathfrak{R})}{dx} + \frac{dd(S - V\mathfrak{S})}{dx^2} - \text{etc.} = 0,$$

$$(R - V\mathfrak{R}) - \frac{d(S - V\mathfrak{S})}{dx} + \text{etc.} = 0$$

etc.

COROLLARY 5

In this case $V = 0$, but hence only those terms involving the quantity V can be thrown out. For, where its differential occurs, since $\frac{dV}{dx} = L$, for L one has to write the value it has for $x = a$, which might not vanish in this case, which is also to be noted on the following differentials:

$$\frac{ddV}{dx^2} = \frac{dL}{dx}, \quad \frac{d^3V}{dx^3} = \frac{ddL}{dx^2} \quad \text{etc.},$$

which values must be taken in general in such a way first before one puts $x = a$ in them.

COROLLARY 6

But if also the first values of y are arbitrary, then these equations must be satisfied by putting $x = 0$, where the same is to be noted as just above. Of course, these equations must be expanded completely before one sets $x = 0$ in them. But in these conditions only the constants entering into the indefinite relation among x and y are determined.

PROBLEM 3

If the function Z, aside from the quantities x, y, p, q, r etc. also involves the two integral formulas $\Phi = \int \mathfrak{Z} dx$ and $\Phi' = \int \mathfrak{Z}' dx$ somehow, so that

$$dZ = Ld\Phi + L'd\Phi' + Mdx + Ndy + Pdp + Qdq + Rdr + etc.,$$

but in these formulas Φ and Φ' the functions $\mathfrak{Z}$ and $\mathfrak{Z}'$ are just determined by the quantities x, y, p, q, r etc. so that

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.,$$
$$d\mathfrak{Z}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + etc.,$$

to define the relation among x and y that this integral formula $\int Zdx$ extended from the limit $x = 0$ to the limit $x = a$ has the maximum or minimum value.

Therefore, one has to define the differential variation of the formula $\int Zdx$, since which is $\delta \int Zdx = \int \delta Zdx$, first we have:

$$\delta Z = L\delta\Phi + L'\delta\Phi' + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

but further

$$\delta\Phi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx \quad \text{and} \quad \delta\Phi' = \int \delta\mathfrak{Z}'dx$$

and hence

$$\begin{aligned} \delta\mathfrak{Z} &= \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}, \\ \delta\mathfrak{Z}' &= \mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}, \end{aligned}$$

from which we conclude

$$\begin{aligned} \delta\Phi &= \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right), \\ \delta\Phi' &= \int dx \left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right). \end{aligned}$$

Therefore, since the differential variation in question is

$$\delta \int Zdx = \int Ldx\delta\Phi + \int L'dx\delta\Phi' + \int Ndx\delta y + \int Pdx\delta p + \text{etc.},$$

let us put $\int Ldx = V$ and $\int L'dx = V'$, and, as above, it will be

$$\begin{aligned} \int Ldx\delta\Phi = V &\int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right) \\ &- \int Vdx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right), \\ \int L'dx\delta\Phi' = V' &\int dx \left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right) \\ &- \int V'dx \left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r - \text{etc.}\right). \end{aligned}$$

But let us put that these integrals $V = \int Ldx$ and $V' = \int L'dx$ are taken in such a way that they vanish for $x = a$, and the first parts of the preceding formulas will immediately become zero, since their values must be taken for the limits $x = a$. Therefore, combining all parts, we will obtain

$$\delta \int Zdx = \int dxy(N - V\mathfrak{N} - V'\mathfrak{N}'),$$

$$+\int dx\delta p(P - V\mathfrak{P} - V'\mathfrak{P}'),$$
$$+\int dx\delta q(Q - V\mathfrak{Q} - V'\mathfrak{Q}')$$
$$+\int dx\delta r(R - V\mathfrak{R} - V'\mathfrak{R}')$$

etc.

But since

$$\int Pdx\delta p = P\delta y - \int \delta y dP,$$
$$\int Qdx\delta q = Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}dQ + \int \frac{\delta y}{dx}ddQ,$$
$$\int Rdx\delta r = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2}dR + \frac{\delta y}{dx^2}ddR - \int \frac{\delta y}{dx^2}d^3R$$

etc.,

we will find the differential variation in question

$$\begin{aligned}
&\delta \int Zdx \\
&= \int dx\delta y \left((N - V\mathfrak{N} - V'\mathfrak{N}') - \frac{d(P - V\mathfrak{P} - V'\mathfrak{P}')}{dx} + \frac{dd(Q - V\mathfrak{Q} - V'\mathfrak{Q}')}{dx^2} - \text{etc.} \right) \\
&+ \delta y \left((P - V\mathfrak{P} - V'\mathfrak{P}') - \frac{d(Q - V\mathfrak{Q} - V'\mathfrak{Q}')}{dx} + \text{etc.} \right) \\
&+ \frac{d\delta y}{dx} \left((Q - V\mathfrak{Q} - V'\mathfrak{Q}') - \frac{d(R - V\mathfrak{R} - V'\mathfrak{R}')}{dx} + \text{etc.} \right) \\
&+ \frac{dd\delta y}{dx^2} \left((R - V\mathfrak{R} - V'\mathfrak{R}') - \text{etc.} \right) \\
&+ \text{etc.}
\end{aligned}$$

COROLLARY 1

For the sake of brevity let us put:

$$N - V\mathfrak{N} - V'\mathfrak{N}' = (N),\; P - V\mathfrak{P} - V'\mathfrak{P}' = (P),\; Q - V\mathfrak{Q} - V'\mathfrak{Q}' = (Q) \text{ etc.},$$

and the indefinite relation among x and y will be expressed by this expression:

$$(N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} = 0,$$

which nevertheless already involves the prescribed limit $x = a$, since the integral formulas $V = \int Ldx$ and $V' = \int L'dx$ were taken in such a way that they vanish for $x = a$.

COROLLARY 2

But the integration of this equation, if it was a differential equation, involves two arbitrary constants, and if also those are arbitrary, then, in order for the formula $\int Zdx$ to obtain a maximal or minimal value, they must be defined in such a way that so for $x = 0$ as for $x = a$ also these equations are satisfied

$$(P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} = 0, \quad (Q) - \frac{d(R)}{dx} + \text{etc.} = 0, \quad (R) - \text{etc.} = 0.$$

COROLLARY 3

If the function Z does not only involve two integral formulas $\Phi = \int \mathfrak{Z}dx$, $\Phi' = \int \mathfrak{Z}'dx$ but also $\Phi'' = \int \mathfrak{Z}''dx$, $\Phi''' = \int \mathfrak{Z}'''dx$ but in such a way that the letters $\mathfrak{Z}$, $\mathfrak{Z}'$, $\mathfrak{Z}''$ etc. denote the functions only of the quantities x, y, p, q, r etc. and they do not involve any more integral formulas, from the solution of the problem the differential variations of formulas of this kind are easily assigned.

PROBLEM 4

If the function Z, aside from the quantities x, y, p, q, r etc. also involves the integral formula $\Phi = \int \mathfrak{Z}dx$ somehow, that

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + etc.,$$

but the function $\mathfrak{Z}$, aside from x, y, p, q, r etc., again involves an integral formula $\Phi = \int \mathfrak{z}dx$, so that

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.,$$

the function $\mathfrak{z}$ on the other hand only consists of the quantities x, y, p, q, r etc., where

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + etc.,$$

to define the relation among x and y that this integral formula $\int Zdx$ extended from $x = 0$ to $x = a$ has a maximum or minimum value.

Solution

For this aim one has to find the differential variation of the formula $\int Zdx$; since $\delta \int Zdx = \int \delta Zdx$, first we will have:

$$\delta Z = L\delta\Phi + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

and hence the differential variation will be

$$\delta \int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.}$$

But now, because of $\delta\Phi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ and

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\Phi + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.},$$

in like manner we will have

$$\delta\Phi = \int \mathfrak{L}dx\delta\Phi + \int \mathfrak{N}dx\delta y + \int \mathfrak{P}dx\delta p + \int \mathfrak{Q}dx\delta q + \int \mathfrak{R}dx\delta r + \text{etc.}$$

Finally, $\delta\Phi = \delta \int \mathfrak{z}dx = \int \delta\mathfrak{z}dx$, and hence, because of

$$\delta\mathfrak{z} = \mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.},$$

it will be

$$\delta\Phi = \int \mathfrak{n}dx\delta y + \int \mathfrak{p}dx\delta p + \int \mathfrak{q}dx\delta q + \int \mathfrak{r}dx\delta r + \text{etc.}$$

Now let $\int \mathfrak{L}dx = v$, and it will be

$$\begin{aligned}\int \mathfrak{L}dx\delta\Phi = v \int dx\,(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})\\ - \int vdx\,(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})\,,\end{aligned}$$

whence we get

$$\delta\Phi = v\int dx\,(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})$$
$$+\int dx\delta y\,(\mathfrak{N} - v\mathfrak{n}) + \int dx\delta p\,(\mathfrak{P} - v\mathfrak{p}) + \int dx\delta q\,(\mathfrak{Q} - v\mathfrak{q}) + \text{etc.}$$

Further, let us set $\int Ldx = V$ and $\int Lvdx = T$, and it will be

$$\int Ldx\delta\Phi = T\int dx\,(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})$$
$$-\int Tdx\,(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})$$
$$+V\int dx\delta y(\mathfrak{N} - v\mathfrak{n}) + V\int dx\delta p(\mathfrak{P} - v\mathfrak{p}) + V\int dx\delta q(\mathfrak{Q} - v\mathfrak{q}) + \text{etc.}$$
$$-\int Vdx\delta y(\mathfrak{N} - v\mathfrak{n}) + \int Vdx\delta p(\mathfrak{P} - r\mathfrak{p}) + \int Vdx\delta q(\mathfrak{Q} - v\mathfrak{q}) + \text{etc.}$$

Therefore, adding all these, the differential variation in question will be

$$\delta\int Zdx = T\int dx(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.})$$
$$+V\int dx\delta y(\mathfrak{N} - v\mathfrak{n}) + V\int dx\delta p(\mathfrak{P} - v\mathfrak{p}) + V\int dx\delta q(\mathfrak{Q} - v\mathfrak{q}) + \text{etc.}$$
$$+V\int dx\delta y(N - V\mathfrak{N} + Vv\mathfrak{n} - T\mathfrak{n})$$
$$+\int dx\delta p(P - V\mathfrak{P} + Vv\mathfrak{p} - T\mathfrak{p})$$
$$+\int dx\delta q(Q - V\mathfrak{Q} + Vv\mathfrak{q} - T\mathfrak{q})$$
$$\text{etc.,}$$

since which must be extended to the limit $x = a$, let us put the integrals $\int Ldx = V$ and $\int Ldx \int \mathfrak{L}dx = T$, since the determination of the integration is arbitrary, to be taken in such a way that they vanish for $x = a$, in order to simplify our expression. Then, for the sake of brevity let us set:

$$N - V\mathfrak{N} - (Vv - T)\mathfrak{n} = (N),$$
$$P - V\mathfrak{P} - (Vv - T)\mathfrak{p} = (P),$$
$$Q - V\mathfrak{Q} - (Vv - T)\mathfrak{q} = (Q),$$
$$R - V\mathfrak{R} - (Vv - T)\mathfrak{r} = (R)$$
$$\text{etc.,}$$

while, as we assumed, $v = \int \mathfrak{L} dx$, and the differential variation in question will be reduced to this form:

$$\delta \int Z dx = \int dx \delta y \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} \right)$$
$$+ \delta y \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} \right)$$
$$+ \frac{d\delta y}{dx} \left((Q) - \frac{d(R)}{dx} + \text{etc.} \right)$$
$$+ \frac{dd\delta y}{dx^2} \left((R) - \text{etc.} \right)$$

etc.

Corollary 1

Since $v = \int \mathfrak{L} dx$, it will be

$$Vv = \int L dx \int \mathfrak{L} dx \text{ and } Vv - T = \int L dx \int \mathfrak{L} dx - \int L dx \int \mathfrak{L} dx = \int \mathfrak{L} dx \int L dx.$$

But since by the assumed determinations the expression $Vv - T$ vanishes for $x = a$, if we put $\int L dx = V$ and $\int \mathfrak{L} V dx = \mathfrak{V}$, these two integrals must be taken in such a way that they vanish for $x = a$.

Corollary 2

Therefore, having introduced these formulas $\int L dx = V$ and $\int \mathfrak{L} V dx = \mathfrak{V}$ into the calculation, it will be

$$N - V\mathfrak{N} + \mathfrak{V}\mathfrak{n} = (N),$$
$$P - V\mathfrak{P} + \mathfrak{V}\mathfrak{p} = (P),$$
$$Q - V\mathfrak{Q} + \mathfrak{V}\mathfrak{q} = (Q),$$
$$R - V\mathfrak{R} + \mathfrak{V}\mathfrak{r} = (R)$$

etc.,

and the differential variation will be expressed by the letters (N), (P), (Q) in the same way as it had been defined above in the first case by the letters N, P, Q etc.

Corollary 3

From these it is now easily concluded, if also the function $\mathfrak{z}$ involves a new integral formula, how the differential variation is expressed then; if it was

$$d\mathfrak{z} = \mathfrak{l}d\Phi' + \mathfrak{m}dx + \text{etc.},$$

then one would additionally have the third formula $\mathfrak{v} = \int \mathfrak{l}\mathfrak{V}dx$ in addition to $V = \int Ldx$ and $\mathfrak{V} = \int \mathfrak{L}Vdx$; everything else should be clear for the attentive reader.

Problem 5

If the function Z, aside from the quantities x, y, p, q, r etc., also involves the integral formula $\Phi = \int \mathfrak{Z}dx$ somehow, that

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + etc.,$$

but the function $\mathfrak{Z}$, aside from the quantities x, y, p, q, r etc., again involves the same integral formula $\Phi = \int \mathfrak{Z}dx$, so that

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.,$$

to define the relation among x and y that the integral formula $\int Zdx$ extended from the limit $x = 0$ to the given limit $x = a$ has a maximal or minimal value.

Solution

As up to this point the differential variation is

$$\delta \int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

but further we have $\delta\Phi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ and

$$\mathfrak{Z} = \mathfrak{L}\Phi + \mathfrak{N}y + \mathfrak{P}p + \mathfrak{Q}dq + \mathfrak{R}r + \text{etc.}$$

But since $\Phi = \int \mathfrak{Z}dx$, it will be

$$\mathfrak{Z} = \frac{d\Phi}{dx} \quad \text{and} \quad \delta\mathfrak{Z} = \frac{\delta d\Phi}{dx} = \frac{d\delta\Phi}{dx}.$$

Let us put

$$\delta\Phi = u \quad \text{and} \quad \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} = w,$$

so that one obtains this equation

$$\frac{du}{dx} = \mathfrak{L}u + w,$$

whose integral, having taken e for the number whose logarithm is $= 1$, reads

$$\mathrm{e}^{-\int \mathfrak{L}dx} u = \int \mathrm{e}^{-\mathfrak{L}dx} w dx,$$

and hence

$$\delta\Phi = \mathrm{e}^{\int \mathfrak{L}dx} \int \mathrm{e}^{-\int \mathfrak{L}dx} dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} \right),$$

whence one concludes

$$\int L dx \delta\Phi = \int \mathrm{e}^{\int \mathfrak{L}dx} L dx \int \mathrm{e}^{-\int \mathfrak{L}dx} dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} \right).$$

Now put $\int \mathrm{e}^{\int \mathfrak{L}dx} L dx = V$, which integral is to be taken in such a way that it vanishes for $x = a$, and let $\mathrm{e}^{-\int \mathfrak{L}dx} V = U$, it will be

$$\int L dx \delta\Phi = -\int U dx \left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} \right);$$

if to which part all remaining parts are added and the above reductions are done, the differential variation in question $\delta \int Z dx$ will result as

$$\begin{aligned} &\delta \int Z dx \\ &= \int dx \delta y \left((N - U\mathfrak{N}) - \frac{d(P - U\mathfrak{P})}{dx} + \frac{dd(Q - U\mathfrak{Q})}{dx^2} - \frac{d^3(R - \mathfrak{U}\mathfrak{R})}{dx^3} + \text{etc.} \right) \\ &+ \delta y \left((P - U\mathfrak{P}) - \frac{d(Q - U\mathfrak{Q})}{dx} + \frac{dd(R - U\mathfrak{R})}{dx^2} - \text{etc.} \right) \\ &+ \frac{d\delta y}{dx} \left((Q - U\mathfrak{Q}) - \frac{d(R - U\mathfrak{R})}{dx} + \text{etc.} \right) \\ &+ \frac{dd\delta y}{dx^2} \left((R - U\mathfrak{R}) - \text{etc.} \right), \end{aligned}$$

from which as above the relation among x and y is found so that the integral formula $\int Zdx$ has the maximal or minimal value for $x = a$; for, this relation is expressed by this equation:

$$(N - U\mathfrak{N}) - \frac{d(P - U\mathfrak{P})}{dx} + \frac{dd(Q - U\mathfrak{Q})}{dx^2} - \frac{d^3(R - U\mathfrak{R})}{dx^3} + \text{etc.} = 0.$$

But then for the determination of the constants introduced by integrations each absolute part can be rendered equal to zero for the case $x = a$ and the case $x = 0$.

COROLLARY

Since we put $e^{-\int \mathfrak{L}dx} V = U$, it will be $V = e^{\int \mathfrak{L}dx} U$ whence by differentiating

$$dV = e^{\int \mathfrak{L}dx}(dU + U\mathfrak{L}dx).$$

But since $dV = e^{\int \mathfrak{L}dx} Ldx$, one will have this differential equation

$$dU + U\mathfrak{L}dx = Ldx,$$

from which the quantity U must be determined in such a way that it vanishes for $x = a$.

PROBLEM 6

If the function Z, aside from the quantities x, y, p, q, r etc., also involves the integral formula $\Phi = \int Zdx$ so that

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + etc.,$$

to define the relation among x and y that this formula $\int Zdx$ has a maximum or minimum value, if extended from the limit $x = 0$ to $x = a$.

Solution

Since the differential variation is

$$\delta \int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

one will also have $\delta\Phi = \int \delta Zdx$, whence by differentiating:

$$d\delta\Phi = Ldx\delta\Phi + Ndx\delta y + Pdx\delta p + Qdx\delta q + Rdx\delta r + \text{etc.},$$

and hence, as before, one finds

$$\delta\Phi = \mathrm{e}^{\int Ldx} \int \mathrm{e}^{-\int Ldx} dx \left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),$$

whence, because of $\int \mathrm{e}^{\int Ldx} Ldx = \mathrm{e}^{\int Ldx}$, we obtain

$$\begin{aligned}\int Ldx\delta\Phi = {} & \mathrm{e}^{\int Ldx} \int \mathrm{e}^{-\int Ldx} dx \left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right) \\ & - \int dx \left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),\end{aligned}$$

which last term is cancelled by the remaining parts. Hence, if we put $\mathrm{e}^{-\int Ldx} = T$, the whole differential variation will be

$$\begin{aligned}\delta \int Zdx = \frac{1}{T} \int dx\delta y & \left(TN - \frac{d.TP}{dx} + \frac{dd.TQ}{dx^2} - \frac{d^3.TR}{dx^3} + \text{etc.}\right) \\ & + \delta y \left(TP - \frac{d.TQ}{dx} + \frac{dd.TR}{dx^2} - \text{etc.}\right) \\ & + \frac{d\delta y}{dx} \left(TQ - \frac{d.TR}{dx} + \text{etc.}\right) \\ & + \frac{dd\delta y}{dx^2} \left(TR - \text{etc.}\right) \\ & \text{etc.}\end{aligned}$$

Therefore, for the formula $\int Zdx$ to become maximal or minimal, the indefinite relation among x and y will be expressed by this equation:

$$TN - \frac{d.TP}{dx} + \frac{dd.TQ}{dx^2} - \frac{d^3.TR}{dx^3} + \text{etc.} = 0;$$

but each absolute part will serve for the determination of the constants introduced by integration.

SCHOLIUM

Therefore, since this analysis does not involve any geometrical considerations, we not only obtained the solutions of all problems extending to the method of maxima and minima which I already gave in my book on maxima and minima, but this method also gave a peculiar determination of the constants, which were completely undetermined in the first method; hence innumerable interesting problems can be resolved quickly, to which the first method can not conveniently be applied. As if, e.g., among all lines to be drawn from a given point not to another point but to another given line or curve the one is required, along which a body descending from that initial point reaches that other curve in the shortest amount of time, this problem is easily solved by consideration of those absolute parts, while in these that condition is prescribed that the curve in question is normal to the given one. But before I end this paper, I want to add an extraordinary theorem for the analysists to examine, whose truth is easily seen from the foundations constituted up to this point and which seems to be immensely useful in integral calculus.

THEOREM

Having propounded the differential Zdx, in which Z is an arbitrary function of the quantities x, y, $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$ etc., and having differentiated it let

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

so that this differential formula Zdx not only contains the first differentials but even others of higher order, then it is easy to decide whether this integral formula admits an integration or is a complete differential or not. For, consider the following expression for constant dx

$$V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.},$$

if which is find equal to zero, the formula Zdx will be integrable; but if not $V = 0$, it will not be integrable.